Indian Statistical Institute
B. Math. (Hons.) I Year
Mid-Semestral Examination II Semester 2008-2009

Date:2-3-2009 Algebra II Instructor: J Biswas

Attempt all questions. Total Marks: 50

1. Let S_n be the group of all permutations of the set $\{1, 2, \cdots, n\}$ and let $\sigma \in S_n$. Describe the permutation matrix M_σ attached to σ. Prove that the map $\sigma \mapsto M_\sigma$ gives a homomorphism of groups $S_n \to GL_n(\mathbb{Z})$.

   (5+5 marks)

2. Let p be a prime integer and $F = \mathbb{F}_p$ be the finite field with p elements.

   a)What is the cardinality of a vectorspace V of dimension n over F?

   b)What is the total number of bases of the above V?

   c)Let $M_n(F)$ be the set of all $n \times n$ matrices with entries from F and let $det : M_n(F) \to F$ be the map which associates to a matrix it's determinant. Show that this map is onto.

   d)Prove that all nonzero values of the det map above is taken the same number of times. What is this number?

   (3+4+3+5 marks)

3. a) Describe all real vector subspaces of $\mathbb{R}^3$ and justify your answer as to why these are all the subspaces.

   b) Prove the set of all polynomials in one variable x with real coefficients is not a finite dimensional real vector space.

   c)True or False: The set of all non-invertible $n \times n$ matrices over any field F is a vector subpsace of all $n \times n$ matrices over F.

   d)Let $A \in M_n(\mathbb{C})$ be a $n \times n$ hermitian matrix, i.e. $a_{ij} = \overline{a_{ji}}$ for all i, j. Show that the hermitian matrices form a real vector space, find a basis for the space and determine it's dimension. (4+4+3+6 marks)

4. Let A be a $m \times n$ matrix and B a $n \times m$ matrix over a field F such that $AB = I_m$ and $BA = I_n$. Prove that $n = m$. (8 marks)